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On Consistency Relations for Cubic Splines-on-Splines and Asymptotic Error Estimates

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1. INTRODUCTION

In the present paper we consider the spline-on-spline technique for calculating the derivative of a function from its values on a uniform mesh. There is computational evidence that this yields better results than the traditional process using a single spline [1]. Dolezal and Tewarson [2] have recently obtained error bounds for spline-on-spline interpolation.

The aim of this paper is to derive new consistency relations between a cubic spline and a cubic spline-on-spline interpolant of its first derivative, and to furnish asymptotic error estimates for the interpolation. For any integer $n \geqslant 1$, let Δ_n: $0 = x_0 < x_1 < \cdots < x_n = 1$ denote a uniform partition of $I = [0, 1]$ with knots $x_i = ih$.

Given a sufficiently smooth function $f(x)$ defined on I, let s be an interpolatory cubic spline of f and p a cubic spline-on-spline interpolant of s' defined by

$$\begin{aligned} &\text{(i)} && s_i = f_i, && i = 0(1)n, \\ &\text{(ii)} && p_i = s'_i, && i = 0(1)n \end{aligned} \tag{1}$$

where $s_i = s(ih)$ and $s'_i = s'(ih)$. Then the following appraisal of the "discretization error" can be found in [2]:

$$|f''_i - p'_i| \leqslant (1/60) \sum_{j=1}^{2} 3^{2-j} \|f^{(4+j)}\| h^{2+j} + \cdots, \qquad i = 0(1)n. \tag{2}$$

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In the present paper we shall derive a sharp asymptotic error estimate:

$$\begin{aligned} &\text{(i)} \qquad f_i'' - s_i'' = (h^2/12) f_i^{(4)} - (h^4/360) f_i^{(6)} + O(h^6), \\ &\text{(ii)} \qquad f_i'' - p_i' = (h^4/90) f_i^{(6)} - (h^6/756) f_i^{(8)} + O(h^7) \end{aligned} \tag{3}$$

for any mesh point x_i distinct from the endpoints.

Using the asymptotic expansion (3)(i), Richardson-type extrapolation gives an $O(h^4)$ second derivate estimate without recourse to the cubic spline-on-spline technique:

$$f_i'' - (1/3)\{4s_{h/2}''(x_i) - s_h''(x_i)\} = (h^4/1440) f_i^{(6)} + \cdots \tag{4}$$

for any mesh point x_i distinct from the endpoints, where $s_h(x)$ and $s_{h/2}(x)$ are cubic spline interpolants of f with uniform mesh sizes h and $h/2$, respectively. On the other hand

$$f_i'' - p_{h/2}'(x_i) = (h^4/1440) f_i^{(6)} + \cdots \tag{5}$$

for any mesh point x_i distinct from the endpoints.

Since the principal parts of the asymptotic expansions (4) and (5) are the same, the cubic spline-on-spline interpolation gives about the same estimate as the extrapolation method. As for computational effort, we have to solve two linear systems of orders n and $2n$ to determine s_h and $s_{h/2}$ in the extrapolation. In the spline-on-spline technique, the coefficient matrices for determining $s_{h/2}$ and $p_{h/2}$ are exactly the same and so $p_{h/2}$ is determined with little additional effort. Hence we are justified using the cubic spline-on-spline technique instead of the extrapolation method.

2. Consistency Relations and Asymptotic Error Estimates

Since s and p depend upon $n+3$ parameters, two additional conditions (which are usually taken near the endpoints) are required for the determination of the splines s and p. For choices of these conditions, see Table 1 in [4].

Here we take them to be the homogeneous end ones:

$$\begin{aligned} &\text{(i)} \qquad \Delta^r s_0' = \nabla^r s_n' = 0, \\ &\text{(ii)} \qquad \Delta^r p_0' = \nabla^r p_n' = 0 \end{aligned} \tag{6}$$

where r is a nonnegative integer and Δ (∇) is the forward (backward) difference operator. By repeated use of the consistency relation

$$(1/6)(s_{i+1}' + 4s_i' + s_{i-1}') = (1/2h)(s_{i+1} - s_{i-1}), \tag{7}$$

the above end condition (6)(i) may be rewritten as

$$s_0' + a_r s_1' = L_r(s_0, s_1, \cdots, s_r), \qquad r \neq 2 \tag{8}$$

where a_r is a rational number and $L_r(s_0, s_1, \cdots, s_r)$ is a linear combination of s_i, $i = O(1)r$.

For a_r, by a simple calculation we have

$$\begin{aligned} (a_0, a_1, a_3) &= (0, -1, 5), \\ a_{r+1} &= (5a_r - 1)/(a_r + 1), \qquad r = 3, 4, \cdots, \\ \lim_{r\to\infty} a_r &= 2 + \sqrt{3}. \end{aligned} \tag{9}$$

We remark that the coefficients for determining s_i' and p_i' are exactly the same under the end conditions (6).

Now we prove the following consistency relation:

THEOREM 1. *Let p be a cubic spline-on-spline interpolant of the derivate of a cubic spline s. Then*

$$\begin{aligned} &(1/36)(p_{i+2}' + 8p_{i+1}' + 18p_i' + 8p_{i-1}' + p_{i-2}') \\ &\quad = (1/4h^2)(s_{i+2} - 2s_i + s_{i-2}) \qquad i = 2(1)n-2. \end{aligned} \tag{10}$$

Proof. By making use of the consistency relation (7) and interpolation condition (1)(ii), we have

$$\begin{aligned} &(1/6)(p_{i+1}' + 4p_i' + p_{i-1}') \\ &\quad = (1/2h)(p_{i+1} - p_{i-1}) \\ &\quad = (1/2h)(s_{i+1}' - s_{i-1}'), \qquad i = 1(1)n-1. \end{aligned} \tag{11}$$

Since $p_{i+2}' + 8p_{i+1}' + 18p_i' + 8p_{i-1}' + p_{i-2}' = (p_{i+2}' + 4p_{i+1}' + p_i') + 4(p_{i+1}' + 4p_i' + p_{i-1}') + (p_i' + 4p_{i-1}' + p_{i-2}')$, by (11) and (7), we have the desired relation.

Next we shall prove the following asymptotic error estimates:

THEOREM 2. *Under the end conditions* (6), *let p be a cubic spline-on-spline interpolant of s'. Then*

$$\begin{aligned} &\text{(i)} \quad f_i'' - s_i'' = (h^2/12)f_i^{(4)} - (h^4/360)f_i^{(6)} + O(h^{\min(6, r-1)}), \\ &\text{(ii)} \quad f_i'' - p_i' = (h^4/90)f_i^{(6)} - (h^6/756)f_i^{(8)} + O(h^{\min(7, r-1)}), \qquad i = 0(1)n. \end{aligned} \tag{12}$$

Kershaw's technique [3] gives

COROLLARY. *For any integer* $r \geqslant 0$, *we have the above asymptotic expansions with* $O(h^6)$ *and* $O(h^7)$ *instead of* $O(h^{\min(6, r-1)})$ *and* $O(h^{\min(7, r-1)})$, *respectively, for any mesh point* x_i *distinct from the endpoints.*

Proof of Theorem 2. First we prove the asymptotic expansion (12)(ii). Since $s_i = f_i$, $i = 0(1)n$, by virtue of (7) we have

$$\begin{aligned}(1/6)(s'_{i+1} + 4s'_i + s'_{i-1}) \\ = f'_i + (h^2/6)f_i^{(3)} + (h^4/120)f_i^{(5)} + (h^6/5040)f_i^{(7)} + \cdots, \qquad i = 1(1)n-1. \end{aligned} \tag{13}$$

Denoting $e(x) = f(x) - (h^4/180)f^{(4)}(x) + (h^6/1512)f^{(6)}(x) - s(x)$ by Taylor series expansion we have

$$\begin{aligned} &\text{(i)} \quad \Delta^r e'_0 = O(h^r), \\ &\text{(ii)} \quad (1/6)(e'_{i+1} + 4e'_i + e'_{i-1}) = O(h^8), \qquad i = 1(1)n-1, \\ &\text{(iii)} \quad \nabla^r e'_n = O(h^r). \end{aligned} \tag{14}$$

By repeated use of (14)(ii), conditions (14)(i) and (iii) can be rewritten

$$\begin{aligned} &\text{for } r \neq 2: \\ &e'_0 + a_r e'_1 = O(h^{\min(8, r)}), \qquad e'_n + a_r e'_{n-1} = O(h^{\min(8, r)}), \\ &\text{for } r = 2: \\ &e'_1 = O(h^2), \qquad e'_{n-1} = O(h^2). \end{aligned} \tag{15}$$

By applying a similar argument [4, 5] to a system of linear equations (15) and (14)(ii), we have

$$e'_i = O(h^{\min(8, r)}), \qquad i = 0(1)n, \tag{16}$$

i.e.,

$$f'_i - s'_i = (h^4/180)f_i^{(5)} - (h^6/1512)f_i^{(7)} + O(h^{\min(8, r)}), \qquad i = 0(1)n. \tag{17}$$

From (11) and (17) we have

$$\begin{aligned}(1/6)(p'_{i+1} + 4p'_i + p'_{i-1}) = f''_i + (h^2/6)f_i^{(4)} + (h^4/360)f_i^{(6)} \\ - (h^6/15120)f_i^{(8)} + O(h^{\min(7, r-1)}), \qquad i = 1(1)n-1. \end{aligned} \tag{18}$$

Using again a similar argument for a system of equations (18), together with (6)(ii), gives

$$f''_i - p'_i = (h^4/90)f_i^{(6)} - (h^6/756)f_i^{(8)} + O(h^{\min(7, r-1)}), \qquad i = 0(1)n. \tag{19}$$

Next we prove the asymptotic expansion (12)(i). The following consistency relations at the endpoints are well known:

$$\begin{aligned} &\text{(i)} && (1/3)(2s_0'' + s_1'') = (2/h^2)(s_1 - s_0) - (2/h)s_0', \\ &\text{(ii)} && (1/3)(2s_n'' + s_{n-1}'') = (2/h^2)(s_n - s_{n-1}) + (2/h)s_n'. \end{aligned} \tag{20}$$

Letting $e(x) = f(x) - (h^2/12)f^{(2)}(x) + (h^4/360)f^{(4)}(x) - s(x)$, by (17) ($i = 0$ and n) and (20) we have

$$\begin{aligned} &\text{(i)} && (1/3)(2e_0'' + e_1'') = O(h^{\min(6, r-1)}), \\ &\text{(ii)} && (1/6)(e_{i+1}'' + 4e_i'' + e_{i-1}'') = O(h^6), \qquad i = 1(1)n-1, \\ &\text{(iii)} && (1/3)(2e_n'' + e_{n-1}'') = O(h^{\min(6, r-1)}). \end{aligned} \tag{21}$$

Using again a similar argument yields the desired asymptotic expansion (12)(i).

Now let us denote by q a cubic spline-on-spline interpolant of the derivate of the cubic spline p. Then, as in the proof of Theorem 1, we have

$$\begin{aligned} &(1/216)(q_{i+3}' + 12q_{i+2}' + 51q_{i+1}' + 88q_i' + 51q_{i-1}' + 12q_{i-2}' + q_{i-3}') \\ &\quad = (1/8h^3)(f_{i+3} - 3f_{i+3} + 3f_{i-1} - f_{i-3}), \qquad i = 3(1)n-3. \end{aligned} \tag{22}$$

Also, as in the proof of Theorem 2, we have

THEOREM 3. *If $\Delta^r q_0' = \nabla^r q_n' = 0$, then*

$$f_i^{(3)} - q_i' = (h^4/60)f_i^{(7)} + O(h^{\min(6, r-2)}), \qquad i = 0(1)n. \tag{23}$$

In addition,

$$f_i^{(3)} - q_i' = (h^4/60)f_i^{(7)} + \cdots \tag{24}$$

for any mesh point x_1 distinct from the endpoints.

By combining the Corollary of Theorem 2 and Theorem 3, we obtain a uniform norm estimate

$$\max_{\alpha \leqslant x \leqslant \beta} |f''(x) - q(x)| = O(h^4), \qquad h \to 0, \tag{25}$$

where α and β $(0 < \alpha < \beta < 1)$ are constant independent of h.

TABLE I

x	e^x $k_2(x)$	e^x $k_3(x)$	e^{5x} $k_2(x)$	e^{5x} $k_3(x)$
1/4	1.00	1.05	1.00	1.03
1/2	1.00	1.00	1.00	1.00
3/4	1.00	1.07	0.99	1.13

3. Numerical Examples

The results of some computational experiments are given in Table I for the functions e^x and e^{5x}. We choose $n = 32$ and

$$\Delta^6 s_0' = \nabla^6 s_n' = 0, \qquad \Delta^6 p_0' = \nabla^6 p_n' = 0 \qquad \Delta^6 q_0' = \nabla^6 q_n' = 0. \tag{26}$$

Let

$$\begin{aligned} k_2(x) &= \{f''(x) - p'(x)\}/\{(h^4/90)f^{(6)}(x)\}, \\ k_3(x) &= \{f^{(3)}(x) - q'(x)\}/\{(h^4/60)f^{(7)}(x)\}. \end{aligned} \tag{27}$$

Then, by (12)(ii) and (24), $k_2(x)$ and $k_3(x)$ tend to 1 for any fixed mesh point $x \in (\alpha, \beta)$ as $h \to 0$.

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